UNIT -II

Definition: Limits:

Let 'f' to be a function defined at all points z in some Neighbourhood of z_0, except possibly for the point z_0 it self.

The limit of $f(z)$ as z approaches z_0 is a number w_0.

$$\lim_{z \to z_0} f(z) = w_0 \rightarrow (1)$$

be made arblitraruty close to w_0 if we choose z choose enough to z_0.

In other words, for each positive number t There is a positive numbers S.

such That

$|f(z) - w_0| < t$ when never $0 < |z - z_0| < S$ →②

Geometerically:

This definition say That for each t-neighbourhood $|w - w_0| < t$ of w_0, There is a S- neighbourhood $|z - z_0| < S$ of z_0 such That The image w of all points z in The S-neighbourhood lie in The t-neighbourhood.

1. Show That if $f(z) = \frac{i}{2}$ where $f(z) = \frac{iz}{2}$ in The open disc $|z| < 1$ and $z = 1$ being on The boundary

Sol

when z in The region $|z| < 1$

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \left| \frac{z-1}{2} \right|$$

which is true of $0 < \left| \frac{z-1}{2} \right| < S$

$\Rightarrow 0 < |z-1| < 2t$

i.e) If we choose $S = t$ we get The

It means that the point $w = f(z)$ can be made arblitraulty close to w_0 if we choose z choose enough to z_0.

In other words, for each positive number t there is a positive number S.

such that

$$|f(z) - w_0| < t \text{ when never } 0 < |z - z_0| < S$$

Geometerically:

This definition say that for each t-neigh bourhood $|w - w_0| < t$ of w_0, there is a S-neighbourhood $|z - z_0| < S$ of z_0 such that the image w of all points z in the S-neigh bourhood lie in the t-neigh bourhood.

1. Show that if $f(z) = \frac{i}{2}$ where $f(z) = \frac{iz}{2}$ in the open disc $|z| < 1$ and $z = 1$ being on the boundary

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$$\Rightarrow 0 < |z-1| < 2t$$

i.e) If we choose $S = t$ we get the

Note:
when limit of a function $f(z)$ exists at a point z_0, it must be unique

Theorem:
Suppose that $f(z) = u(x,y) + i v(x,y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$

Then

$$\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} \rightarrow (1)$$

$$u(x,y) = u_0 \rightarrow (2)$$

and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 \rightarrow (2)$

Proof:

Assume that statement (1) is true

prove that (2) is true

By definition of limit for each t there is a positive number such that

$|(u-u_0) + i(v-v_0)| < t$ whenever

$0 < |(x-x_0) + i(y-y_0)| < s$

Now;

$|u-u_0| \leq |(u-u_0) + i(v-v_0)| < t$

$|v-v_0| \leq |(u-u_0) + i(v-v_0)| < t$

$|u-u_0| < t$, $|v-v_0| < t$

whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < s$

② is true.

Now, suppose ② is true

prove That ① is true.

For each positive numbers ϵ Then exists a fine numbers δ_1 and δ_2 such that

$|u - u_0| < \epsilon/2$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_1$

and

$|v_1 - v_0| < \epsilon/2$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$

Consider $|(u+iv) - (u_0+iv_0)| = |(u+v_0)+i(v-v_0)|$

$\leq |u-u_0| + |v_0 - v_0|$

$\leq \epsilon/2 + \epsilon/2 = \epsilon$

$\therefore |(u+iv) - (u_0+iv_0)| < \epsilon$ whenever $0 < |(x+iy) - (x_0+iy_0)| < \delta$

$\therefore$ ① is true.

Continuity:

The function $f(z)$ is continuity at a point z_0, if

(i) $\lim_{z \to z_0} f(z)$ exists.

(ii) $f(z_0)$ exists.

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

Condition (iii) says that for each positive number ϵ there is a number δ, such that

$|f(z) - f(z_0)| < \epsilon$ whenever

$|z - z_0| < \delta$

A function of a complex variable is said to be continous in a region R if it continious at each point in R.

property:
(i) sum of two continous function (ii) product continous function is a continous function
(iii) Quotient of two continous function is continous where the dinomenator not equal to zero.
(iv) polynomial is continous in the entire plane.

Definition: Derivatives.

Let f be a function whose d of definition contaious a ~~neighbourh~~ heighbour of a point z_0. The derivation of at z_0 denoted by $f'(z_0)$ is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided the limit exists. The function of f is said to be differentiable at z_0 when its derivative at z_0 exists.

example:

suppose that $f(z) = z^2$, find $f'(z)$ at any point z

Sol

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{(z+\Delta z)^2 - z^2}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left| \frac{\cancel{z^2} + (\Delta z)^2 + 2z\Delta z - \cancel{z^2}}{\Delta z} \right|$$

$$= \lim_{\Delta z \to 0} \left| \frac{\cancel{\Delta z}(\Delta z + 2z)}{\cancel{\Delta z}} \right|$$

$$= \lim_{\Delta z \to 0} \left[\Delta z + 2z \right]$$

Let $\Delta z = 0$

$$= 2z$$

$$\boxed{f'(z) = 2z}$$

example: 2

$$f(z) = |z|^2 = z\bar{z}$$

$$u + iv = (x+iy)(x-iy)$$

$$u + iv = x^2 + y^2$$

$$u = x^2 + y^2$$

$$v = 0.$$

(i) eventhough, partial derivative of u & v exists, The function is not differentiable

(ii) $f(z) = |z|^2$ The function is continous at each point in the plane since its components u and v are polynomials which are continous at each point.

∴ Continuity of a of a function at a point does not implie the existance of a derivative There

(iv) But existance of The derivative of a function at a point gives The continousty of the function at The point.

10 mark

The cauchy - Rieman Equations.

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be differentiable at a point $z_0 = x_0 + iy_0$. Then $u(x,y)$ and $v(x,y)$ have first order partial derivatives $u_x(x_0, y_0)$, $u_y(x_0, y_0)$, $v_x(x_0, y_0)$ and $v_y(x_0, y_0)$ at (x_0, y_0) and these partial derivatives satisfy The Cauchy - Rieman equation.

(CR equations) given by.

$u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0)$
$= -v_x(x_0, y_0)$

Also $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0 y_0)$
$= v_y(x_0, y_0) - i u_y(x_0, y_0)$

proof:

Since $f(z) = u(x, y) + i v(x, y)$ is difference at $z_0 = x_0 + i y_0$

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists}$$

and hence.

The limit is independent of the path in which h approaches zero. $h = h_1 + i h_2$.

Let $h = h_1 + i h_2$ $\rightarrow$ (1)

Now, $\frac{f(z_0+h) - f(z_0)}{h}$

$= \frac{u(x_0+h_1, y_0+h_2) + i v(x_0+h_1, y_0+h_2)}{h+ih_2} \frac{- u(x_0, y_0) - i v(x_0, y_0)}{h+ih_2}$

$= \left[\frac{u(x_0+h_1, y_0+h_2) - u(x_0, y_0)}{h_1+ih_2} \right] + i \left[\frac{v(x_0+h_1, y_0+h_2) - v(x_0, y_0)}{h_1+ih_2} \right]$

Suppose $h \to 0$ along the real axis so that $h = h_1$

Then $f'(z_0) = \lim_{h_1 \to 0} \left[\frac{f(z_0+h_1) - f(z_0)}{h_1} \right]$

$$= \lim_{h_1 \to 0} \left[\frac{u(x_0+h_1, y_0) - u(x_0, y_0)}{h_1} \right]$$

$$+ i \lim_{h_1 \to 0} \left[\frac{v(x_0+h_1, y_0) - v(x_0, y_0)}{h_1} \right]$$

$$= u_x(x_0, y_0) + i v_x(x_0, y_0) \to (1)$$

Now suppose $h \to 0$ along the imaginary axis so that $h = ih_2$.

$$\therefore f'(z_0) = \lim_{ih_2 \to 0} \left[\frac{f(z_0+h_2) - f(z_0)}{ih_2} \right]$$

$$= \lim_{h_2 \to 0} \left[\frac{u(x_0, y_0+h_2) - u(x_0 y_0)}{ih_2} \right]$$

$$+ i \lim_{h_2 \to 0} \left[\frac{v(x_0, y_0+h_2) - v(x_0, y_0)}{ih_2} \right]$$

$$= \left(\frac{u_y(x_0, y_0)}{i} \right) + i \left[\frac{v_y(x_0 y_0)}{i} \right]$$

$$= \frac{1}{i} u_y(x_0, y_0) + v_y(x_0, y_0)$$

$$= -iu_y(x_0, y_0) + v_y(x_0 y_0) \to (2)$$

from (1) & (2) we get

$$f'(z_0) = u_x(x_0 y_0) + iv_x(x_0, y_0)$$

$$= v_y(x_0, y_0) - iu_y(x_0, y_0)$$

Equating Real and Imaginary parts, we get.

$$u_x(x_0, y_0) = v_y(x_0, y_0) \qquad u_x = v_y$$
$$u_y(x_0, y_0) = -v_x(x_0, y_0) \qquad -u_y = v_x$$

Remark 1: Since $f'(z) = u_x + i v_x = v_y - i u_y$

we have

$$|f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2$$

Also $$|f'(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2$$

Further.
$$|f'(z)|^2 = u_x v_y - u_y v_x$$
$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$\boxed{|f'(z)|^2 = \frac{\partial(u,v)}{\partial(x,y)}}$$

Remark 2:

The Cauchy-Riemann equations provide a necessary condition for differentiability at a point. Hence if the C.R equations are not satisfied for a complex function at any point then we can conclude that the function is not differentiable

For example, consider the function

$f(z) = \bar{z} = x - iy$

Here $u(x,y) = x$ and $v(x,y) = -y$

$\therefore u_x(x,y) = 1$ and $v_y(x,y) = -1$

$u_x \neq v_y$ so that C.R equations are not satisfied at any poit z.

Here the function $f(z) = \bar{z}$ is now here differentiable.

Remark: C.R equations in polar coordinates.

Come Sary

Theorem:

Let $f(z) = u(r,\theta) + iv(r,\theta)$ be differentiable at $z = re^{i\theta} \neq 0$

Then, $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r} \frac{\partial u}{\partial \theta}$

Further $f'(z) = \frac{r}{z}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$

proof: we have $x = r\cos\theta$ and $y = r\sin\theta$

Hence $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$

$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$

Also $\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$

$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$

$$\therefore \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta$$

$$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \quad \text{[using C.R equation}$$

$$= \frac{\partial u}{\partial r} \text{ (using ①]}$$

Thus $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\left(\dfrac{\partial v}{\partial \theta}\right)$

Now $r\left(\dfrac{\partial u}{\partial r} + i\dfrac{\partial v}{\partial r}\right) = r\left[\left(\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r}\right) + i\left(\dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial r}\right)\right]$

$$= r\left[\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right]$$

$$= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial y}\right)$$

$$= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + iy\left(\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y}\right)$$

$$= x f'(z) + iy f'(z)$$

$$= \cancel{(x+iy) f'(z)}$$

$$= z f'(z)$$

$$\boxed{\therefore f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)}$$

we now prossed to express C.R equations in get amther from.

Let $f(z) = u(x,y) + iv(x,y)$

Since $x = \dfrac{z+\bar{z}}{2}$ and $y = \dfrac{z-\bar{z}}{2}$ we have

$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z+\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z+\bar{z}}{2}\right)$

Thus f can be thought of as a function of z and $\bar{z}$. Though z and $\bar{z}$ are not independent variable we form the partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ as if z and $\bar{z}$ are independent variables with this convention we have the following Theorem.

U.Q. 5 mark.

Prove that the function $|x|$ is continuous at $x=0$ but not differentiable at $x=0$

$f(x) = |x|$

Sol Now $f(0) = |0| = 0$

$$f(0+0) = \lim_{h\to 0} |0+h| = 0$$

$$f(0-0) = \lim_{h\to 0} |0-h| = 0$$

$\therefore f(x)$ is continuous at $x=0$

But

$$\frac{f(x)-f(0)}{x-0} = \frac{x-0}{x-0} = \frac{x}{x} = 1 \text{ if } (x>0$$

$$\frac{f(x)-f(0)}{x-0} = \frac{-x-0}{x-0} = -\frac{x}{x} = -1 \text{ if} (x$$

$\therefore \lim_{h\to 0} \frac{f(x)-f(0)}{x-0}$ does not exists.

Thus if does not have a derivatives at $x=0$, eventhough f is continous at 0.

check whether

CR equation satisfied not

2 mark. $f(z) = \bar{z}$

Sol

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$f(z) = \bar{z} = x - iy$$

$$u(x,y) + iv(x,y) = x - iy$$

$$u(x,y) = x$$

$$v(x,y) = -y$$

$u(x,y) = x$	$v(x,y) = y$
$\frac{\partial u}{\partial x} = 1$	$\frac{\partial v}{\partial x} = 0$
$\frac{\partial u}{\partial y} = 0$	$\frac{\partial v}{\partial y} = -1$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = 0 = -0 = -\frac{\partial v}{\partial x}$$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ CR equation not satisfied.

$\therefore$ The function is not differentiable.

② CR. satisfy are not $f(z)=z^3$

Sol

$f(z)=z^3$ $\quad a^3+b^3+3a^2b+3b^2a$

$z=x+iy$

$z^3=(x+iy)^3$

$z^3=x^3+(iy)^3+3x^2iy+3x(iy)^2$

$=x^3-iy^3+i3x^2y-3xy^2$

$z^3=(x^3-3xy^2)+i(3x^2y-y^3)$

$f(z)=z^3$

$u(x,y)+iv(x,y)=(x^3-3xy^2)+i(3x^2y-y^3)$

$u(x,y)=x^3-3xy^2$

$\frac{\partial u}{\partial x}=3x^2-3y^2$

$\frac{\partial u}{\partial y}=-6xy$

$v=3x^2y-y^3$

$\frac{\partial v}{\partial x}=6xy$

$\frac{\partial v}{\partial y}=3x^2-3y^2$

$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y}=-6xy=-\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\therefore$ CR Equation is satisfies

The function is differentiable

(3). CR satisfies are not $f(z) = \operatorname{Re} z$.

Sol

$f(z) = \operatorname{Re} z$ $\quad z = x+iy$

$f(z) = \operatorname{Re} z$

$z = x + iy$

$\operatorname{Re} = x$

$u + iv = x$.

$u = x$ | $v = 0$

$\frac{\partial u}{\partial x} = 1$ $\quad u_x = 1$ | $\frac{\partial v}{\partial x} = 0$ $\quad v_x = 0$

$u_x = 1$ $\quad v_x = 0$

$u_y = 0$ $\quad v_y = 0$

$\frac{\partial u}{\partial y} = 0$ $\quad u_y = 0$ | $\frac{\partial v}{\partial y} = 0$ $\quad v_y = 0$

$u_x \neq u_y$

$u_x \neq v_y$.

$u_y = -v_x$ CR does not nowhere differentiam.

Theorem. 1. If $f(z)$ is a differentiable function the C.R equations can be put in the form $\frac{\partial f}{\partial \bar z} = 0$

Proof.

$$\frac{\partial f}{\partial \bar z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar z}$$

$$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Thus $\frac{\partial f}{\partial \bar z} = 0 \iff \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ which is the complex form of the C.R equation (Refer Theorem 2.7).

Thus the C.R equation ... be put in the form $\frac{\partial f}{\partial \bar{z}} = 0$.

Prob: 1.

Prove that $f(z) = \begin{cases} \frac{z \operatorname{Re} z}{|z|} & \text{if } \ldots \\ \ldots & \text{if } \ldots \end{cases}$ is continious at $z=0$ but not differentiable at $z=0$

Sol

First we shall prove that $\lim_{z \to 0}$ ...

Now, $|f(z) - 0| = \left| \frac{z \operatorname{Re} z}{|z|} \right| = |\operatorname{Re} z|$

Further $|\operatorname{Re} z| \leq |z|$

$\therefore$ For any given $\epsilon > 0$ if we choose $\delta = \epsilon$ we get

$|z| = |z - 0| < \delta \Rightarrow |f(z) - 0| < \epsilon$

Hence f is continious at $z=0$

Now, we prove that $f(z)$ is not differentiable at $z=0$

$$\frac{f(z) - f(0)}{z - 0} = \frac{z \operatorname{Re} z}{z|z|} = \frac{\operatorname{Re} z}{|z|}$$

$$= \frac{x}{\sqrt{x^2+y^2}} \quad \text{where } z = x+iy$$

Along the path $y = mx$

$$\frac{f(z) - f(0)}{z - 0} = \frac{x}{\sqrt{x^2+m^2x^2}} = \frac{1}{\sqrt{1+m^2}}$$

The value of the limit depends on m and hence on the path along which $z \to 0$

$\therefore \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$ does not exist

$\therefore f(z)$ is not differentiable at $z = 0$

(2) Prove that $f(z) = z \operatorname{Im} z$ is differentiable only at $z = 0$ and find $f'(0)$

Sol

$$f(z) = z \operatorname{Im} z$$
$$= (x + iy)\, y$$

$\therefore u(x,y) = xy$, and $v(x,y) = y^2$

$u_x = y$, $v_x = 0$, $u_y = x$, $v_y = 2y$

Clearly the C.R. equation are satisfied only at $z = 0$ further all the first order partial derivatives are continuous at $z = 0$

Hence, $f(z)$ is differentiable at $z = 0$

Also $f'(0) = u_x(0,0) + i v_x(0,0) = 0$

(3) Show that $f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

is not differentiable at $z = 0$.

Sol

$$\frac{f(z) - f(0)}{z - 0} = \frac{xy^2(x+iy)}{x^2+y^4} \left(\frac{1}{x+iy}\right)$$

$$= \frac{xy^2}{x^2+y^4}$$

$\therefore$ Along the path $x = my^2$.

$$\frac{f(z) - f(0)}{z - 0} = \frac{my^4}{m^2y^4 + y^4} = \frac{m}{m^2+1}$$

The value of the limit depends on m and hence depends on the path along which $z \to 0$

$\therefore \lim_{z \to 0} \dfrac{f(z)-f(0)}{z-0}$ does not exists

$\therefore f(z)$ is not differentiable at $z=0$.

(7) Prove That The function $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ $f'(0)$ does not exist.

Sol $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Here $u(x,y) = \dfrac{x^3 - y^3}{x^2+y^2}$, and $v(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

if $(x,y) \neq (0,0)$ and $u(0,0) = v(0,0) = 0$

Now $u_x(0,0) = \lim_{h \to 0} \dfrac{u(h,0) - u(0,0)}{h}$

$= \lim_{h \to 0} \left(\dfrac{\frac{h^3}{h^2} - 0}{h} \right) = 1$

similery $u_y(0,0) = -1$, $v_x(0,0) = 1$ and $v_y(0,0) = 1$ (verify) Thus $u_x(0,0) = v_y(0,0) = 1$

and $u_y(0,0) = -v_x(0,0) = -1$ so That

C.R equation are satisfied at $z=0$

Now $\dfrac{f(z) - f(0)}{z-0} = \dfrac{x^3 - y^3}{(x^2+y^2)(x+iy)} + i \dfrac{x^3+y^3}{(x^2+y^2)(x+iy)}$

$= \dfrac{1-m^3}{(1+m^2)(1+m)} + i \dfrac{1+m^3}{(1+m^2)(1+im)}$

Hence the value of the Limit depends on the path along which $z \to 0$.

Thus $\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$ does not exist

Hence f is not differentiable at 0.

5. Prove that $f(z) = \sin x \cosh y + i \cos x \sinh y$ is differentiably at every point.

Sol

$f(z) = \sin x \cosh y + i \cos x \sinh y$

$\therefore u(x,y) = \sin x \cosh y \rightarrow (1)$.

$u = (x,y) = \sin x \cosh y$

$v = (x,y) = \cos x \sinh y$

$u_x = \cos x \cosh y$ and $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ and $v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$ for all x, y

Hence C.R equation are saticfied at every point further all the first order partial derivatices are continous

Hence u $f(z)$ is differentiable at every points.

6. Find continous a and b so that the function $f(z) = a(x^2 - y^2) + ibxy + c$ is differentiable at every point

Sol

Here $u(x,y) = a(x^2 - y^2) + c$ and

$v(x,y) = bxy$

$u_x = 2ax$ $v_x = by$

$u_y = -2ay$ and $v_y = bx$

Clearly $u_x = v_y$ and $u_y = -v_x$ iff $2a = b$

$\therefore$ C-R equation are satisfied at all points iff $2a = b$.

$\therefore$ The function $f(z)$ is differentiable for all values of a, b with $2a = b$.

(7). Show That $f(z) = \sqrt{r}\,(\cos\theta/2 + i\sin\theta/2)$ where $r > 0$ and $0 < \theta < 2\pi$ is differentiable and find $f'(z)$

Sol

$$f(z) = \sqrt{r}\,(\cos\theta/2 + i\sin\theta/2)$$

$u = \sqrt{r}\cos(\theta/2)$ and $v = \sqrt{r}\sin(\theta/2)$

$\therefore \dfrac{\partial u}{\partial r} = \dfrac{1}{2\sqrt{r}}\cos(\theta/2)$ and $\dfrac{\partial v}{\partial r} = \dfrac{1}{2\sqrt{r}}\sin(\theta/2)$

$\dfrac{\partial u}{\partial r} = -\dfrac{\sqrt{r}}{2}\sin(\theta/2)$ and $\dfrac{\partial v}{\partial r} = \dfrac{\sqrt{r}}{2}\cos\frac{\theta}{2}$

Now, $\dfrac{1}{r}\dfrac{\partial v}{\partial\theta} = \dfrac{1}{r}\left(\dfrac{\sqrt{r}}{2}\cos\left(\dfrac{\theta}{2}\right)\right)$

$$= \frac{1}{2\sqrt{r}}\cos(\theta/2)$$

$$= \frac{\partial u}{\partial r}$$

Thus $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial\theta}$

Similarly $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}$

$$= \frac{1}{2\sqrt{r}\sin(\theta/2)}$$

Hence the C-R equations (in polar f

are satisfied Further all the first order partial derivatives are continuous.

Hence $f'(z)$ exists

Also $f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$ (refere therem) Q.8

$= \frac{r}{z}\left[\frac{1}{2\sqrt{r}}\cos(\theta/2) + \frac{i}{2\sqrt{r}}\sin(\theta/2)\right]$

$= \frac{r}{2\sqrt{r}z}(\cos\theta/2 + i\sin\theta/2)$

$= \frac{1}{2z}\left(\sqrt{r}(\cos\theta/2 + i\sin\theta/2)\right)$

$= \left(r(\cos\theta + i\sin\theta)\right)^{1/2}$

By using Demoivres Theorem $= [r(\cos\theta + i\sin\theta)]^{1/2}$

$= \frac{1}{2z}(\sqrt{z}) = \frac{1}{2\sqrt{z}}$

$$\boxed{\text{Hence } f'(z) = \frac{1}{2\sqrt{z}}}$$

H.W. ~~check~~

CR equation are not sufficient for differentiable at a point as shown in the following example.

① . $f(z) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } z \neq 0 \\ & \text{if } z = 0 \end{cases}$

Sol

Given

$u(x,y) + iv(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } z \neq 0 \\ & \text{if } z = 0 \end{cases}$

$u(x,y) = \frac{xy}{x^2+y^2}$ | $v = 0$, $v_x = 0$, $v_y = 0$

$u_x(0,0) = \lim_{h \to 0}\left[\frac{u(h,0) - u(0,0)}{h}\right]$

$= \lim_{h \to 0}\left[\frac{0-0}{h}\right] = 0$

$\boxed{u_x = 0}$

llly we can P.T $u_y(0,0) = 0$

$u_x = 0$ | $v_x = 0$

$u_y = 0$ | $v_y = 0$

$\therefore$ CR equation is satisfied $u_x = v_y,\ u_y = 0$ at $z=0$. Now Along the path

$y = mx$

$$f(z) = \frac{x\,mx}{x^2+m^2x^2} = \frac{mx^2}{m^2(1+m^2)} = \frac{m}{1+m^2},\ x \neq 0$$

Hence $z \to 0$ along the path $y = mx$

$f(z) \Rightarrow \frac{m}{1+m^2}$ which is different for different values of m. Hence $f(z)$ does not have a limit as $z \to 0$ so that $f(z)$ is not continuous at $z=0$

Thus $f(z)$ is not differentiable at $z=0$

2. CR equation are not sufficient for differentiable at a point as shown in following examples. $f(z) = \sqrt{|xy|}$

Sol

$$f(z) = \sqrt{|xy|}$$

$$u(x,y) + i v(x,y) = \sqrt{|xy|}$$

$u(x,y) = \sqrt{|xy|}$ | $v(x,y) = 0$, $v_x = 0$, $v_y = 0$

$$u_x(0,0) = \lim_{h\to 0}\left[\frac{u(h,0) - u(0,0)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{0-0}{h}\right] = 0$$

$u_x(0,0) = 0$ | $v_x = 0$

$v_y = 0$

iiith $u_y(0,0) = 0$.

CR equation satisfied $u_x = v_y$ $u_y = -v_x$ at $z = 0$ Now Along The path $y = mx$

$$\frac{f(z) - f(0)}{z} = \frac{\sqrt{|x\,mx|} - 0}{x + imx} = \frac{x\sqrt{m}}{x(1+im)}$$

$$= \frac{\sqrt{m}}{1+im}.$$

Hence $z \to 0$ along The path $y = mx$

$f(z) = \frac{\sqrt{m}}{1+im}$ which is different for different value of m, Hence $f(z)$ does not have a limit as $z \to 0$ Sofhat $f(z)$ is not continious at $z = 0$ Thus $f(z)$ is not differentiable at $z = 0$

Q. Show That $f(z) = \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

is not different at $z = 0$.

Sol

$$\frac{f(z) - f(0)}{z} = \frac{\frac{xy^2(x+iy)}{x^2+y^4}}{x+iy}$$

$$= \frac{xy^2\cancel{(x+iy)}}{x^2+y^4} \times \frac{1}{\cancel{x+iy}}$$

$$\frac{f(z) - f(0)}{z} = \frac{xy^2}{x^2+y^4}$$

Along The path $x = my^2$

$$\frac{f(z) - f(0)}{z} = \frac{my^2\,y^2}{m^2y^4 + y^4} = \frac{my^4}{y^4(1+m^2)} = \frac{m}{1+m^2}$$

$z = x + iy$.

Proof:

Since $u(x,y)$ and its first order partial derivatives are continous at (x,y) we have by the mean value theorem for functions of two variable.

$u(x+h_1, y+h_2) - u(x,y) = h_1 u_x(x,y) + h_2 u_y(x,y) + h_1 \epsilon_1 + h_2 \epsilon_2 \to ①$

where ϵ_1 and $\epsilon_2 \to 0$ as h_1 and $h_2 \to 0$

similarly

$v(x+h_1, y+h_2) - v(x,y) = h_1 v_x(x,y) + h_2 v_y(x,y) + h_1 \epsilon_3 + h_2 \epsilon_4 + \dots \to ②$

where $\epsilon_3, \epsilon_4 \to 0$ as h_1 and $h_2 \to 0$

Let $h = h_1 + i h_2$

Then $\dfrac{f(z+h) - f(z)}{h} = \dfrac{1}{h}\Big[u(x+h_1, y+h_2) - u(x,y) + i v(x+h_1, y+h_2) - v(x,y)\Big]$

$= \dfrac{1}{h}\Big[\{h_1 u_x(x,y) + h_2 u_y(x,y) + h_1 \epsilon_1 + h_2 \epsilon_2\} + i\{h_1 v_x(x,y) + h_2 v_y(x,y) + h_1 \epsilon_3 + h_2 \epsilon_4\}$

$= \dfrac{1}{h}\Big[h_1\{u_x(x,y) + i v_x(x,y)\} + h_2\{u_y(x,y) + i v_y(x,y)\} + h_1(\epsilon_1 + i\epsilon_3) + h_2(\epsilon_2 + i\epsilon_4)\Big]$

using C.R equation:-

$= \dfrac{1}{h}\Big(h\, u_x(x,y) - i h\, u_y(x,y) + h_1(\epsilon_1 + i\epsilon_3) + h_2(\epsilon_2 + i\epsilon_4)\Big]$

$= u_x(x,y) - i u_y(x,y) + \dfrac{h_1}{h}(\epsilon_1 + i\epsilon_3) + \dfrac{h_2}{h}(\epsilon_2 + i\epsilon_4)$

$\left|\frac{h_1}{h}\right| \leq 1$

NOW, $\left|\frac{h_1}{h}\right| \leq 1$, $\frac{h_1}{h}(\epsilon_1 + i\epsilon_3) \to 0$ as $h \to 0$

$\frac{h_1}{h}(\epsilon_1+i\epsilon_3) \to 0$ as $h \to 0$

Similarly $\frac{h_2}{h}(\epsilon_2 + i\epsilon_4)$

$\therefore \lim_{h \to 0} \frac{f(z+h)-f(z)}{h} = u_x(x,y) - i u_y(x,y)$

Hence f is differentiable.

M.Sc

1.

V.Imp Imp

Theorem:

An analytic function in a region D with its derivative zero at every point of the domain is a constant.

proof: Let $f(z) = u(x,y) + iv(x,y)$ be analytic in D and

$f'(z) = 0 \; \forall \; z \in D$

since $f'(z) = u_x + iv_x$

$f'(z) = v_y + iu_y$

where $u_x = u_y = v_x = v_y = 0$

$u_x=0$, $u_y=0$ / $u_x=0$, $v_x=0$

$u(x,y)$ and $v(x,y)$ are constant function and hence $f(z)$ is constant.

5 Km

V.I

2) An analytic function in a region with constant modulus is constant

Sol: Let $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D since $|f|$ is constant, we have $u^2+v^2 = c$ where c is a constant differentiating partially with respect to x we get

$2uu_x + 2vv_x = 0$

(i.e) $uu_x + vv_x = 0 \to (1)$ D.p.w.r.t.x, $uu_x + vv_x$ / u

$\|^{ly}$ by differentiating partially with respect to y we get. $uu_y + vv_y$

$$uu_y + vv_y = 0 \to (2)$$

using C.R equation in (1) and (2) we get

$$uu_x - vv_y = 0 \to (3) \qquad uu_x - vv_y = 0$$

$$uv_y + vu_x = 0 \to (4) \qquad uv_y + vv_x = 0$$

Eliminating u_y from (3) & (4) we get

$$u^2 u_x - uvv_y = 0$$

$$v^2 u_x + uvv_y = 0$$

$$(u^2 + v^2) u_x = 0$$

$$u^2 + v^2 = c \qquad u_x = 0$$

$\|^{ly}$ we can prove that $v_x = 0$

$$f'(z) = u_x + iv_x = 0.$$

So that Hence f is Constant.

(2) Any analytic function $f(z) = u + iv$ with $\arg f(z)$ constant is itself a constant function.

$\tan^{-1}\left(\frac{v}{u}\right)$

Sol $\arg f(z) = \tan^{-1}\left(\frac{v}{u}\right) = c$ where c is a Constant.

$$\therefore \frac{v}{u} = k \text{ where } k \text{ is a constant}$$

$$\therefore v = ku \qquad v = ku$$

$$v_x = ku_x \text{ and } v_y = ku_y$$

eliminating k from the above equations we get $u_x v_y = v_x u_y$

$$\therefore u_x v_y - u_y v_x = 0.$$

$u u_x + v v_y = 0$ (C.R equation)

$u^2 x + u^2 y = 0$

$\therefore u_x = 0$ and $u_y = 0$

Hence u is constanims. 111^{ly} we can in prove that v is

for Constant.

$f(z) = u + iv$ is Constant.

(3) If $f(z)$ and $f(\bar{z})$ are analytic in a a region D.S.T $f(z)$ is constant in That region.

V.V.I

5 mark.

Soln. Let $f(z) = u(x,y) + iv(x,y)$

$f(\bar{z}) = u(x,y) - iv(x,y)$

$\because f(z)$ is analytic in D we have

$u_x = v_y \quad u_y = -v_x$

$\because f(\bar{z})$ is analytic in D we h

$u_x = -v_y$ & $u_y = -v_x$

Adding we get $u_x = 0 \quad v_y = 0$

$u_y = 0 = u_x$

$u_x = 0 = v_x$

$u_x = 0 \quad u_y = 0$

$u_y = 0 = u_x$

$u_x = 0 = v_x$

V.V.I $f'(z) = u_x + iv_x = 0$

$f'(z) = u_x + iv_x = 0$

$\therefore f(z)$ is constant in D

(4) P.T the function $f(z)$ and $f(\bar{z})$ are simultaneously analytic.

Thereom

$u_x + iv_x$

Sol suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in a rigeion D

Then the first order partial derivatives of u and v are continous and satisfy the CR equations.

$$u_x = v_y \rightarrow (1) \quad v_x = -u_y \rightarrow (2)$$

Now $\overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$

$$\overline{f(\bar{z})} = u_1(x,y_1) + iv(x,y)$$

[where $u_1(x,y) = u(x,-y)$
$v(x,y) = -v(x,-y)$

$\frac{\partial V}{\partial x} = \frac{\partial u}{\partial x}$

Hence

$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial x}$

$$\frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v_1}{\partial y} = \frac{\partial v_1}{\partial y} \quad (\text{use } (1))$$

$u_y = -v_x$

$$\frac{\partial u_1}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v_1}{\partial x} = \frac{\partial v_1}{\partial x}$$

v_x $\frac{\partial V}{\partial x}$

∴ The first order partial derivatures of u_1 and v_1 are continuous and satisfy the CR equation in D.

The $\overline{f(\bar{z})}$ is analytic in D Then

∴ $f(\bar{z})$ is analytic in D.

$f(z)$ is also analytic

(5) Test whether the following functions are anaylytic.

(1) $z^3 + z$.

Sol

$z^3 + z$

$$f(z) = z^3 + z$$

$$u[x,y] + iv(x,y) = (x+iy)^3 + (x+iy)$$

$$u + iv = (x^3 - iy^3 + 3x^2iy - 3xy^2) + x + iy$$

$$u + iv = (x^3 - 3x^2y + x) + i(-y^3 + 3xy + y)$$

$u = x^3 - 3xy^2 + x$

$u_x = 3x^2 - 3y^2 + 1$

$u_y = -6xy$

$v_x = -y^3 + 3x^2y + y$

$v_x = 6xy$

$v_y = -3y^2 + 3x^2 + 1$

$$u_x = v_y \quad u_y = -v_x$$

∴ The CR equation is satisfy

∴ $z^3 + z$ is analytic.

Q. If $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$ P.T $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4\partial^2}{\partial z \partial \bar{z}}$

Sol

$z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$

$x = \frac{z + \bar{z}}{2}$

$z - \bar{z} = 2iy$

$y = \frac{z - \bar{z}}{2i}$

$x = \frac{z + \bar{z}}{2}$ $\quad$ $y = \frac{z - \bar{z}}{2i}$

$\frac{\partial x}{\partial z} = 1/2$ $\quad$ $\frac{\partial y}{\partial z} = \frac{1}{2i}$

$\frac{\partial x}{\partial \bar{z}} = 1/2$ $\quad$ $\frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$

$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$

$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left(\frac{+1}{2i}\right)$

$= \frac{1}{2}\frac{\partial}{\partial x} + \partial/\partial y\left(\frac{+i}{2i^2}\right)$

$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$

$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \partial/\partial y \frac{\partial y}{\partial z}$

$= \partial/\partial x\left(1/2\right) + \partial/\partial y\left(\frac{1}{2i}\right)$

$$= \frac{1}{2} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \left(\frac{i}{2i^2} \right)$$

$$\frac{\partial}{\partial z} = \frac{1}{2} \partial/\partial x - \frac{1}{2} i \frac{\partial}{\partial y} \quad (2)$$

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z} \left(\frac{\partial}{\partial \bar{z}} \right)$$

$$= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \frac{1}{2} \left(\frac{\partial}{\partial x} + i \partial/\partial y \right)$$

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \left(\partial^2/\partial x^2 + \frac{\partial^2}{\partial y^2} \right)$$